## Fem-fenics

 $General\ purpose\ Finite\ Element\ library\\ for\ GNU-Octave$ 

WORK IN PROGRESS (HELP AND REMARKS ARE WELCOME)

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December 2, 2013

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# Chapter 1

## Introduction

Fem-fenics is an open source package (pkg) for the resolution of partial differential equations with Octave. The project has been developed during the Google Summer of Code 2013 with the help and the sustain of the GNU-Octave community under the supervision of prof. De Falco.

The report is structured as follows:

- in chapter 2 we provide a simple reference guide for beginners
- in chapter 3 is presented a detailed explanation of the relevant parts of the program. In this way, the interested reader can see what there is "behind" and expecially anyone interested in it can learn quickly how it is possible to extend the code and contribute to the project.
- in chapter 4 more examples are provided. For a lot of them, we present the octave script alongside with the code for Fenics (in C++ and/or Python) in order to provide the user with a quick reference guide.

If you think that going inside the report could be boring, it is available a wiki at

http://wiki.octave.org/Fem-fenics

while if you want to see how the project has grown during the time you can give a look at

http://gedeone-gsoc.blogspot.com/

Finally, the API is available as Appendix but also at the following address

http://octave.sourceforge.net/fem-fenics/overview.html

# Chapter 2

## Introduction to Fem-fenics

#### 2.1 Installation

Fem-fenics is an external package for Octave, which means that it can be installed only once that Octave has been successfully installed on the PC. Furthermore, as Fem-fenics is based on Fenics, it is also needed a running version of the latter. They can be easily installed following the guidelines provided on the official Octave [1] and Fenics [2] websites. Once that Octave and Fenics are correctly installed, to install Fem-fenics open Octave (which now is provided with a new amazing GUI) and type

```
>> pkg install fem-fenics -forge
```

That's all! For any problem during the installation don't hesitate to contact us. To be sure that everything is working fine, load the fem-fenics pkg and run one of the examples provided within the package:

>> pkg load fem-fenics
>> femfenics\_examples()

For a description of the examples, look at chapter 4.

**NOTE** For completing the installation process successfully, the form compiler FFC and the header file dolfin.h should also be available on the machine. They are managed automatically by Fenics if it is installed as a binary package or with Dorsal. If it has been done manually, please be sure that they are available before starting the installation of Fem-fenics.

## 2.2 General layout and first example

A generic problem has to be solved in two steps:

1. a .ufl file where the abstract problem is described: this file has to be written in Unified Form Language (UFL), which is a domain specific language for defining discrete variational forms and functionals in a notation close to pen-and-paper formulation. UFL is easy to learn, and the User manual provides explanations and examples [3].

2. a script file .m where the abstract problem is imported and a specific problem is implemented and solved: this is the script file where the femfenics functions described in the following chapters are used.

We provide immediately a simple example in order to familiarize the user with the code.

The Poisson equation In this example, we show how it is possible to solve the Poisson equation with mixed Boundary Conditions. If we indicate with  $\Omega$  the domain and with  $\Gamma = \Gamma_N \cup \Gamma_D$  the boundaries, the problem can be expressed as

$$\Delta u = f$$
 on  $\Omega$  
$$u = 0$$
 on  $\Gamma_D$  
$$\nabla u \cdot n = g$$
 on  $\Gamma_N$ 

where f, g are data which represent the source and the flux of the scalar variable u. A possible variational formulation of the problem is: find  $u \in H^1_{0,\Gamma_D}$ :

$$\begin{split} a(u,v) &= L(v) \qquad \forall v \in H^1_{0,\Gamma_D} \\ a(u,v) &= \int_{\Omega} \nabla u \cdot \nabla v \\ L(v) &= \int_{\Omega} f v + \int_{\Gamma_N} g v \end{split}$$

The abstract problem can thus be written in the Poisson.ufl file immediately. The only thing that has to be specified at this stage is the space of Finite Elements used for the discretization of  $H^1_{0,\Gamma_D}$ . In this example, we choose the space of continuous lagrangian polynomial of degree one

```
FiniteElement("Lagrange", triangle, 1)
```

but many more possibilities are available.

```
element = FiniteElement("Lagrange", triangle, 1)

u = TrialFunction(element)

v = TestFunction(element)

f = Coefficient(element)

g = Coefficient(element)

a = inner(grad(u), grad(v))*dx

L = f*v*dx + g*v*ds
```

It is always a good idea to check if the ufl code is correctly written before importing it into Octave. Typing

```
>> ffc -l dolfin Poisson.ufl
```

in the shell shouldn't produce any error.

We can now implement and solve a specific instance of the Poisson problem with Octave. The parameters are setted as follow

- $\Omega = [0,1] \times [0,1]$
- $\Gamma_D = (0, y) \cup (1, y) \subset \partial \Omega$
- $\Gamma_N = (x,0) \cup (x,1) \subset \partial \Omega$
- $f = 10 \exp \frac{(x 0.5)^2 + (y 0.5)^2}{0.02}$
- $q = \sin(5x)$

As a first thing we need to load into Octave the pkgs previously installed

```
pkg load fem-fenics msh
```

The ufl file can thus be imported inside Octave. For every specific element defined inside the ufl file there is a specific function which stores it for later use

- ufl\_import\_FunctionSpace ('Poisson') is a function which looks for the finite element space defined inside the file called Poisson.ufl; if everything is ok, it generates a function which we will use later
- ufl\_import\_BilinearForm ('Poisson') is a function which looks for the rhs of the equation, i.e. for the bilinear form defined inside Poisson.ufl
- ufl\_import\_LinearForm ('Poisson') is a function which looks for the linear form.

In some cases one could be interested in using these functions separately but if, as in our example, all the three elements are defined in the same uff file (and only in this case), the import\_ufl\_Problem ('Poisson') can be used, which generates at once all the three functions described above

```
ufl_import_Problem ('Poisson');
```

To set the concrete elements which define the problem, the first things to do is to create a mesh. It can be managed easily using the msh pkg. For a structured squared mesh

```
x = y = linspace (0, 1, 33);
msho = msh2m_structured_mesh (x, y, 1, 1:4);
```

Once that the mesh is available, we can thus initialize the Fem-fenics mesh using the function Mesh ():

```
mesh = Mesh (msho);
```

To initialize the functional space, we have to specify as argument only the fem-fenics mesh, because the finite element type and the polynomial degree have yet been specified in the ufl file:

```
V = FunctionSpace('Poisson', mesh);
```

Essential BC can now be applied using DirichletBC (); this function receives as argument the functional space, a function handle which specifies the value to set, and the label of the sides where the BC applies. In this case, homogenous boundary conditions hold on the left and right side of the square

```
bc = DirichletBC(V, @(x, y) 0.0, [2; 4]);
```

The last thing to do before solving the problem, is to set the coefficients specified in the uff file. To set them, the function Expression () can be used passing as argument a string which specifies the name of the coefficient (it is important that they are called in the same way as in the uff file: the source term 'f' and the normal flux 'g'), and a function handle with the value prescribed:

```
ff = Expression ('f',
    @(x,y) 10*exp(-((x - 0.5)^2 + (y - 0.5)^2) / 0.02));
gg = Expression ('g', @(x,y) sin (5.0 * x));
```

Another possibility for dealing with the coefficients defined in the ufl file would have been to use the function Constant () or Function (). The coefficients can thus be used together with the FunctionSpace to set the Bilinear and the Linear form

```
a = BilinearForm ('Poisson', V, V);
L = LinearForm ('Poisson', V, ff, gg);
```

The discretized representation of our operator is obtained using the functions assemble () or  $assemble\_system$  (), which also allow to specify the BC(s) to apply

```
[A, b] = assemble_system (a, L, bc);
```

Here A is a sparse matrix and b is a column vector. All the functionalities available within Octave can now be exploited to solve the linear system. The easisest possibility is the backslash command:

```
u = A \setminus b;
```

Once that the solution has been obtained, the  ${\tt u}$  vector is converted into a Femfenics function and plotted  ${\tt plot}$  () or saved  ${\tt save}$  () in the vtu format

```
u = Function ('u', V, sol);
save (u, 'poisson')
plot (u);
```

The complete code for the Poisson problem is reported below, while in figure 2.1 the output is presented.

```
#load the pkg and import the ufl problem

pkg load fem-fenics msh

import_ufl_Problem ('Poisson')

# Create the mesh and define function space

x = y = linspace (0, 1, 33);
```

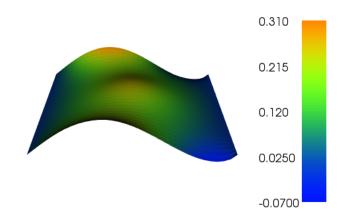


Figure 2.1: The result for the Poisson equation

```
mesh = Mesh(msh2m_structured_mesh (x, y, 1, 1:4));
   V = FunctionSpace('Poisson', mesh);
   # Define boundary condition and source term
10
   bc = DirichletBC(V, Q(x, y) 0.0, [2;4]);
   ff = Expression ('f', Q(x,y) 10*exp(-((x - 0.5)^2 + (y - 0.5)^2) /
       0.02));
   gg = Expression ('g', @(x,y) \sin (5.0 * x));
   #Create the Bilinear and the Linear form
15
   a = BilinearForm ('Poisson', V, V);
L = LinearForm ('Poisson', V, ff, gg);
   #Extract the matrix and compute the solution
19
[A, b] = assemble_system (a, L, bc);
sol = A \setminus b;
u = Function ('u', V, sol);
^{24}\, # Save solution in VTK format and plot it
save (u, 'poisson')
26 plot (u);
```

## Chapter 3

# Implementation

Fem-fenics aims to fill a gap in Octave: even if there are packages for the creation of mesh [4], for the postprocessing of data [5] and for the resolution of some specific pde [6] [7], no general purpose finite element library is available.

The goal of the project is thus to provide a package which can be used to solve user defined problems and which is able to exploit the functionality provided with Octave.

Instead of writing a library from scratch, an interface to one of the finite element library which are already available has been created. Among the many libraries taken into account, the one which was best suited for our purposes seemed to be the FEniCS project. It "is a collection of free, open source, software components with the common goal to enable automated solution of pde." In particular, Dolfin is the C++/Python interface of FEniCS, providing a consistent Problem Solving Environment for ODE and PDE. The idea has been to create wrappers in Octave for C++ Dolfin, in a similar way to what it has been done for Python. This is a very natural choice, because Octave is mainly written in script language and in C++. It is in fact possible to implement an Octave interpreter function in C++ through the native oct-file interface or, conversely, to use Octave's Matrix/Array Classes in a C++ application [8].

The works can be summirized as follows (fig. 3.1):

the elements already available in Octave for the resolution of PDE (Mesh and Linear Algebra) have been exploited, and wrappers to the other FEniCS functions added. To allow exchanges between this programs, the necessary functions for converting an Octave mesh/matrix into a FEniCS one and viceversa have been written.

Two main ideas have guided us throughout the realization of the pkg:

- keep the syntax as close as possible to the original one in Fenics (Python)
- make the interface as simple as possible.

## 3.1 General layout of a class

Seven new classes are implemented for dealing with FEniCS objects and for using them inside Octave:

• boundarycondition stores and builds a dolfin::DirichletBC

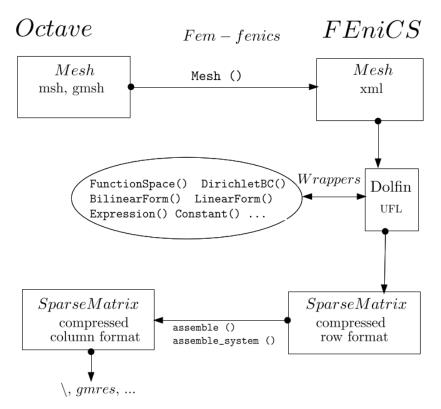


Figure 3.1: General layout of the package

- coefficient stores an expression object which is used for the evaluation of user defined values
- expression is needed for internal use only as explained below
- form stores a general dolfin::Form and can be used either for a dolfn::BilinearForm as well as for a dolfin::LinearForm
- function for the dolfin::Function objects
- functionspace stores the user defined FunctionSpace
- mesh converts a PDE-tool like mesh structure in a dolfin::Mesh

The classes are written with the "usual" C++ style, but they need to be derived publicly from octave\_base\_value and to be added to the Octave interpreter [8]. When a type is used for the first time during a session, it is also temporarily registered in the interpreter after all the other basic types (int, double, ...).

The general layout of a class can thus be kept simple and with the main purpose of storing the associated FEniCS objects, which is done throughout boost::shared\_ptr< > to the corresponding FEniCS type. All the classes also implement at least two constructor: a default constructor which is necessary to register a type in the Octave interpreter, and a constructor which takes as argument the corresponding dolfin type.

As an example, the form class implementation follows, while classes which differs from the general layout are presented below in more details.

```
#ifndef _FORM_OCTAVE_
   #define _FORM_OCTAVE_
2
   #include <memory>
   #include <vector>
   #include <dolfin.h>
   #include <octave/oct.h>
   class form : public octave_base_value
    public:
     form () : octave_base_value () {}
14
     form (const dolfin::Form _frm)
16
       : octave_base_value (), frm (new dolfin::Form (_frm)) {}
18
     form (boost::shared_ptr <const dolfin::Form> _frm)
19
       : octave_base_value (), frm (_frm) {}
     void
     print (std::ostream& os, bool pr_as_read_syntax = false) const
          os << "Form " << ": is a form of rank " << frm->rank ()
          << " with " << frm->num_coefficients ()
```

```
<< " coefficients" << std::endl;
27
       }
28
     ~form(void) {}
30
     bool is_defined (void) const { return true; }
33
     const dolfin::Form & get_form (void) const { return (*frm); }
35
     const boost::shared_ptr <const dolfin::Form> &
36
     get_pform (void) const { return frm; }
    private:
40
     boost::shared_ptr <const dolfin::Form> frm;
41
     DECLARE_OCTAVE_ALLOCATOR;
43
     DECLARE_OV_TYPEID_FUNCTIONS_AND_DATA;
44
45
   };
46
47
   static bool form_type_loaded = false;
48
49
   DEFINE_OCTAVE_ALLOCATOR (form);
   DEFINE_OV_TYPEID_FUNCTIONS_AND_DATA (form, "form", "form");
51
   #endif
```

#### 3.1.1 Shared pointer

In all the classes presented above, the private members are stored using a boost::shared\_ptr< > to the corresponding FEniCS type. This is done because we have to refer in several places to resources which are built dynamically and we want that it is destroyed only when the last references is destroyed [9]. For example, if we have two different functional spaces in the same problem, like with Navier-Stokes for the velocity and the pressure, the mesh is shared between them and no one has its own copy. Furthermore, they are widely supported inside DOLFIN, and it can thus be avoided to have a copy of the same object for FEniCS and another one for DOLFIN: there is just one copy which is shared among DOLFIN and FEniCS.

#### 3.1.2 The mesh class

In addition to usual methods, the mesh class implemens functionalities which allow to deal with meshes as they are currently available with the msh pkg, i.e. in the (p, e, t) format, and in Fenics, i.e. in the xml Dolfin format. It is therefore necessary to have two different constructors

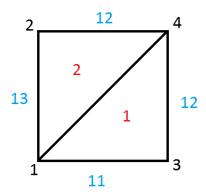


Figure 3.2: The (very) simple mesh for our example

where the first one accepts as input a mesh in (p, e, t) format and convert it into a xml one, while the latter load the mesh stored in the \_filename.xml file.

The constructor are used within the Mesh () function, which therefore accepts as argument either a mesh generated with the msh pkg or a string with the name of the file where the dolfin mesh is stored.

Furthermore, if a mesh is stored in another different format, the program dolfin-convert can try to convert it to the dolfin xml format. For example, for a mesh generated with Metis:

```
Shell: >> dolfin-convert msh.gra msh.xml
```

and then inside the Octave script:

```
mesh = Mesh ('msh.xml');
```

Before exploring the code in more details, the main differences between the two storing formats are presented using the very simple, but rather instructive, example of a unit square mesh with just two elements, fig. 3.2.

**pet** A mesh is represented using the three matrices p, e, t, and, using msh, we can easily obtain the mesh for our example typing

```
mesh = msh2m_structured_mesh ([0 1], [0 1], 1, [11 12 12 13])
```

The matrix p stores information about the coordinates of the vertices

```
>> mesh.p

0 0 1 1 x-coordinates
0 1 0 1 y-coordinates
```

Thus the vertex in the  $n^{th}$  column is labelled as the vertex number n, and so on.

The matrix t stores information about the connectivity

```
1 1 number of the first vertex of the element
3 4 number of the second vertex of the element
4 2 number of the third vertex of the element
0 0
```

The first element is thus the one obtained connecting vertices 1-3-4 and so on. The matrix e stores information related to every side edge, like the number of the vertices of the boundary elements, and the number of the geometrical border containing the edge, which is a convinient way to trate boundary conditions in a problem.

```
>> mesh.e
 1
        2
                   first vertex of the side edge
     3
             1
 3
     4 \quad 4 \quad 2
                   second vertex of the side edge
 0
     0 0
              0
 0
     0
        0
             0
11 12 12
             13
                   label of the geometrical border containing the edge
 0
     0
         0
              0
 1
     1
         1
              1
```

The side edge between vertex 1-3 is labelled 11, between 3-4 is 12...

**dolfin xml** A mesh is an object of the dolfin::Mesh class which stores information only about the coordinates of the vertices (like p) and the information about the connectivity (like t). A mesh can thus be manipulated using the functions and the methods of the class, which are presented below. Instead, the information about boundaries is not directly stored in the mesh. The mesh used in the example is stored as

Conversion between the formats The first necessary step in our way to a package which links Octave and FEniCS is to convert a mesh from the (p, e, t)

format into the dolfin xml one. Furthermore, as dolfin provides methods and functions which allow to manipulate a mesh and which don't have a conterpart in the msh pkg, we have also created wrappers for them (specifically for mesh::refine).

As it has been showed above, the main difference between (p, e, t) and dolfinxml is the way in which the boundaries are distinguished. The former stores all the information in the e matrix, while the latter uses the functions and the methods of the dolfin::mesh class to set/get informations about a mesh. The most useful classes available in dolfin are recalled

 MeshIterator To know whether an edge belongs or not to the boundary, we can iterate over all the edges of our mesh using the classes provided by dolfin:

```
for (dolfin::FacetIterator f (mesh); ! f.end (); ++f)
{
   if ((*f).exterior () == true)
   {
      //do something with the boundary cells
   }
}
```

• MeshFunction To store data related to a mesh, dolfin provides the template class MeshFunctions. "A MeshFunction is a function that can be evaluated at a set of mesh entities. A MeshFunction is discrete and is only defined at the set of mesh entities of a fixed topological dimension. A MeshFunction may for example be used to store a global numbering scheme for the entities of a (parallel) mesh, marking sub domains or boolean markers for mesh refinement." [10] For example, in the function mshm\_refine of the msh package, the list of cells to be refined is stored as a MeshFunction, which for every cell says whether or not it has to be refined:

```
dolfin::CellFunction<bool> cell_markers (mesh);
cell_markers.set_all (false);

for (octave_idx_type i = 0;
    i < cells_to_refine.length (); ++i)
cell_markers.set_value (cells_to_refine (i) , true);</pre>
```

• MeshValueCollection "It differs from the MeshFunction class in two ways. First, data does not need to be associated with all entities (only a subset). Second, data is associated with entities through the corresponding cell index and local entity number (relative to the cell), not by global entity index, which means that data may be stored robustly to file."[11] It is thus obvious that it is better to use the MeshValueCollection whenever saving or writing a mesh.

The containers classes presented above can be used by their own, but to set/get data from a mesh it is better to use the methods provided by the classes:

• MeshDomains "The class MeshDomains stores the division of a Mesh into subdomains. For each topological dimension 0 <= d <= D, where

D is the topological dimension of the Mesh, a set of integer markers are stored for a subset of the entities of dimension d, indicating for each entity in the subset the number of the subdomain. It should be noted that the subset does not need to contain all entities of any given dimension; entities not contained in the subset are "unmarked"." [12]

• MeshData "The class MeshData is a container for auxiliary mesh data, represented either as MeshFunction over topological mesh entities, arrays or maps. Each dataset is identified by a unique user-specified string." [13]

Geometry from (p, e, t) to xml dolfin Converting the vertices and cells from (p, e, t) in the xml format can be done using the dolfin editor, while caution has to be taken for storing information associated with boundaries and subdomains, as presented in the next paragraph.

```
dolfin::MeshEditor editor;
boost::shared_ptr<dolfin::Mesh> msh (new dolfin::Mesh ());
editor.open (*msh, D, D);
editor.init_vertices (p.cols ());
editor.init_cells (t.cols ());
if (D == 2)
 {
   for (uint i = 0; i < p.cols (); ++i)</pre>
     editor.add_vertex (i,
                       p.xelem (0, i),
                       p.xelem (1, i));
   for (uint i = 0; i < t.cols (); ++i)
     editor.add_cell (i,
                     t.xelem(0, i) - 1,
                     t.xelem(1, i) - 1,
                     t.xelem (2, i) - 1);
 }
if (D == 3)
 {
editor.close ();
```

Subdomain markers: from (p, e, t) to dolfin xml There are no fundamental differences between the 2D and 3D case, and they are thus treated together referring to the general dimension D. The subdomain information is contained in the t matrix, and is temporarily copyed to a MeshValueCollection. For every column of the t matrix, i.e. for every element of the mesh, we have to look for the corresponding element in the DOLFIN mesh. We use the class MeshIterator for moving around on the DOLFIN mesh:

```
dolfin::MeshValueCollection<uint> my_cell_marker (D);
```

The all\_vertices\_in\_the\_ith\_column is just like a pseudo code: we have to be sure that the Cell pointed by f is the one corresponding to the  $i^{th}$  column of the matrix, checking one-by-one the vertices:

in 2D the cell is a triangle, and we thus have to check 3 vertices. As we don't know the order in which vertices are visited, we have to check all the 3!=6 different combinations:

where the entities(std::size\_t dim) method returns an array with the indexes of the elements of dimension dim. Thus we use dim = 0 as we are looking for vertices.

In the 3D case, our cell is a tetrahedron, and we have to check all the 4! = 24 possibilities, each of which is composed by 4 assertions; in total we have almost one hundred conditions!

Now that the information is stored in our function, it can be associated to the mesh

```
*(mesh.domains ().markers (D)) = my_marked_cell;
```

Subdomain markers: from dolfin xml to (p, e, t) In the DOLFIN .xml file, the information is stored like:

When the file is read using DOLFIN, the information is automatically associated with the mesh as a MeshValueCollection named cell\_domains, which can be accessed to extract the information using the MeshDomains class. Obviously we have to be sure that the information is available within the file that we are reading, and that it is related to Cell, i.e. to elements of dimension D, before it is associated with the last row of the t matrix:

```
dolfin::MeshFunction<uint> my_cell_marker;
if (! mesh.domains ().is_empty ())
  if (mesh.domains ().num_marked (D) != 0)
    my_cell_marker = *(mesh.domains ().cell_domains ());

for (j = 0; j < t.cols (); ++j)
    t(D + 1, j) = my_cell_marker[j];</pre>
```

**Boundary Markers** For boundary markers, things work in a similar way, as long as we remember that we are working with objects of dimension D - 1. In this case, the main difference is in the .xml file: it is no longer enough to say to what cell element the label is referred to, but we have to specify to which D-1 entity (a side or a face) the label is referred. For example:

The cell number "0" is a triangle, and to the local\_entity number "0", i.e. to the side number "0", is associated the label "12", while to the side number "2" is associated the label "11". To the side number "1", there are no labels associated. The number of the local\_entity refers to the enumeration of the reference element. In any case, it is DOLFIN which takes care of the conversion of indeces from this format to the usual one, and we can thus use methods and functions as explained for the subdomain markers.

Mesh refine Now that it is possible to convert meshes between Octave and DOLFIN, the functions availables in the dolfin::mesh class can be used to improve the functionality of the msh package. For the moment, it has been added the possibility of refining a mesh, either uniformly or specifying the list of the vertices we want to be refined. The function is now part of the msh pkg[4], and a more detailed desciption has been provided previously [14].

#### 3.1.3 The functionspace class

A dolfin::FunctionSpace is defined by specifying a mesh and the type of the finite element which we want to use. The mesh is handled as presented above, while the FE are specified inside the .ufl file. Possible choices are [15]:

Finite Element Space	Symbol
Argyris	ARG *
Arnold-Winther	AW *
Brezzi-Douglas-Marini	BDM
Crouzeix-Raviart	CR
Discontinuous Lagrange	DG
Hermite	HER*
Lagrange	CG
Mardal-Tai-Winther	MTW *
Morley	MOR*
Nédélec 1st kind H (curl)	N1curl
Nédélec 2nd kind H (curl)	N2curl
Raviart-Thomas	RT

where the Finite Elements denoted with  $\ast$  are not yet fully supported inside FEniCS.

### 3.2 General layout of a function

There are three general kind of functions in the code: functions which create an abstract problem (wrappers to UFL), functions which create the specific instance of a problem (wrapper to FEniCS) and functions which discretize the problem and generates the matrices.

## 3.3 Wrappers to UFL

As stated in section 2.2, a problem is divided in two files: a .ufl file where the abstract problem is described in Unified Form Language (UFL), and a script file .m where a specific problem is implemented and solved. We suppose that they are called Poisson.ufl and Poisson.m. In order to use the information stored in the UFL file, i.e. the bilinear and the linear form, they have to be imported inside Octave. When the UFL file is compiled using the ffc compiler, a header file Poisson.h is generated. In this header file, it is defined the Poisson class, which derives from dolfin::Form, and the constructor for the bilinear and linear form are setted. This file is thus available only at compilation time, but it has to be included somehow in the wrapper function for the Bilinear and the Linear form. An easy solution would have been to write a set of pre established problems where the user could only change the values of the coefficient for a specific problem; but, as we want to let the user free to write its own variational problem, a different approach has been adopted. The ufl file is compiled at run time and generates an header file. Then, a Poisson.cc file is written from a template which take as input the name of the header file and is compiled including the Poisson.h file; now the corresponding octave functions for the specific problem is available and will be used from BilinearForm, LinearForm, FunctionSpace, ... . As an example it is presented the import uff BilinearForm

function import\_ufl\_BilinearForm (var\_prob)

```
3
     %the function which writes the var-prob.cc file
     generate_rhs (var_prob);
     %the function which writes the makefile
     generate_makefile (var_prob, private);
     % the makefile is executed in a terminal:
     % 1) generate the header file from ufl
     % ffc -l dolfin var_prob.ufl
     % 2) compile the var_prob.cc
14
     % mkoctfile var_prob.cc -I.
     system (sprintf ("make -f Makefile_%s rhs", var_prob));
16
18
19
   endfunction
20
```

```
function output = generate_rhs (ufl_name)
     STRING ="
3
     #include "@@UFL_NAME@@.h"
6
     DEFUN_DLD (@@UFL_NAME@@_BilinearForm, args, , ""A =
         fem_rhs_@@UFL_NAME@@ (FUNCTIONAL SPACE, COEFF)"")
9
      const functionspace & fspo1
       = static_cast<const functionspace&> (args(0).get_rep ());
       const functionspace & fspo2
14
       = static_cast<const functionspace&> (args(1).get_rep ());
       const dolfin::FunctionSpace & U = fspo1.get_fsp ();
       const dolfin::FunctionSpace & V = fspo2.get_fsp ();
18
       @@UFL_NAME@@::BilinearForm a (U, V);
19
20
21
23
     }":
24
25
     STRING = strrep (STRING, "@@UFL_NAME@@", ufl_name);
26
27
     fid = fopen (sprintf ("%s_BilinearForm.cc", ufl_name), 'w');
28
     fputs (fid, STRING);
29
     output = fclose (fid);
30
31
   endfunction
```

## 3.4 Wrappers to DOLFIN

The general layout of a function is very simple and is composed of 4 steps which we describe using an example:

All the functions presented above follow this general structure, and thus here we present in detail only functions which present some differences.

#### 3.4.1 Sparse Matrices

#### 3.4.2 Polymorphism

#### 3.4.3 DirichletBC and Coefficient

These two functions take as input a function handle which cannot be directly evaluated by a dolfin function to set, respectively, the value on the boundary or the value of the coefficient. It has thus been derived from dolfin::Expression a class "expression" which has as private member an octave function handle and which overloads the function eval(). In this way, an object of the class expression can be initialized throughout a function handle and can be used inside dolfin because "it is" a dolfin::Expression

```
class expression : public dolfin::Expression
   {
2
3
     . . .
5
     eval (dolfin::Array<double>& values,
           const dolfin::Array<double>& x) const
        octave_value_list b;
        b.resize (x.size ());
        for (std::size_t i = 0; i < x.size (); ++i)</pre>
          b(i) = x[i];
        octave_value_list tmp = feval (f->function_value (), b);
         Array<double> res = tmp(0).array_value ();
        for (std::size_t i = 0; i < values.size (); ++i)</pre>
           values[i] = res(i);
```

**DirichletBC** The BC are imposed directly to the mesh setting to zero all the off diagonal elements in the corresponding line. This means that we could loose the symmetry of the matrix, if any. To avoid this problem, instead of the method apply() it is possible to use the function <code>assemble\_system()</code>, which preserves the symmetry of the system but which needs to build together the lhs and the rhs.

Coefficient The coefficient of the variational problem can be specified using either a Coefficient or a Function. They are different objects which behave in different ways: a Coefficient, as exlained above, overloads the eval() method of the dolfin::Expression class and it is evaluated at run time using the octave function feval (). A Function instead doesn't need to be evaluated because it is assembled copying element-by-element the values contained in the input vector.

## 3.5 Wrapper to FEniCS

#### 3.5.1 Code on the fly

# Chapter 4

# More Advanced Examples

# 4.1 Navier-Stokes equation with Chorin-Temam projection algorithm

Navier-Stokes: we learn how to deal with a vector-field problem and how we can save the solution using the fem\_save () function. We also use the msh pkg to generate a mesh using gmesh.

```
# Copyright (C) 2010 Anders Logg
# Define function spaces (P2-P1)
V = VectorElement("CG", triangle, 2)
Q = FiniteElement("CG", triangle, 1)
  # Define trial and test functions
   u = TrialFunction(V)
   v = TestFunction(V)
  # Define coefficients
  k = Constant(triangle)
  u0 = Coefficient(V)
  f = Coefficient(V)
nu = 0.01
# Define bilinear and linear forms
eq = (1/k)*inner(u - u0, v)*dx + inner(grad(u0)*u0, v)*dx + 
    nu*inner(grad(u), grad(v))*dx - inner(f, v)*dx
a = lhs(eq)
L = rhs(eq)
```

```
# Copyright (C) 2010 Anders Logg
# Define function spaces (P2-P1)
V = VectorElement("CG", triangle, 2)
Q = FiniteElement("CG", triangle, 1)
```

```
# Define trial and test functions
p = TrialFunction(Q)

q = TestFunction(Q)

# Define coefficients
k = Constant(triangle)
u1 = Coefficient(V)

# Define bilinear and linear forms
a = inner(grad(p), grad(q))*dx
L = -(1/k)*div(u1)*q*dx
```

```
# Copyright (C) 2010 Anders Logg
  # Define function spaces (P2-P1)
V = VectorElement("CG", triangle, 2)
  Q = FiniteElement("CG", triangle, 1)
   # Define trial and test functions
6
   u = TrialFunction(V)
   v = TestFunction(V)
# Define coefficients
k = Constant(triangle)
u1 = Coefficient(V)
p1 = Coefficient(Q)
14
# Define bilinear and linear forms
a = inner(u, v)*dx
L = inner(u1, v)*dx - k*inner(grad(p1), v)*dx
```

```
pkg load fem-fenics msh
import_ufl_Problem ("TentativeVelocity");
import_ufl_Problem ("VelocityUpdate");
import_ufl_Problem ("PressureUpdate");
# We can either load the mesh from the file as in Dolfin but
# we can also use the msh pkg to generate the L-shape domain
8 L-shape-domain;
mesh = Mesh (msho);
# Define function spaces (P2-P1).
V = FunctionSpace ('VelocityUpdate', mesh);
Q = FunctionSpace ('PressureUpdate', mesh);
14
  # Set parameter values and define coefficients
dt = 0.01;
   T = 3.;
k = Constant ('k', dt);
```

```
f = Constant ('f', [0; 0]);
   u0 = Expression ('u0', @(x,y) [0; 0]);
20
  # Define boundary conditions
noslip = DirichletBC (V, @(x,y) [0; 0], [3, 4]);
outflow = DirichletBC (Q, @(x,y) 0, 2);
   # Assemble matrices
26
a1 = BilinearForm ('TentativeVelocity', V, V, k);
   a2 = BilinearForm ('PressureUpdate', Q, Q);
28
   a3 = BilinearForm ('VelocityUpdate', V, V);
   A1 = assemble (a1, noslip);
   A3 = assemble (a3, noslip);
31
   # Time-stepping
   t = dt; i = 0;
   while t < T
     # Update pressure boundary condition
37
     inflow = DirichletBC (Q, @(x,y) sin(3.0*t), 1);
38
39
     # Compute tentative velocity step
40
     L1 = LinearForm ('TentativeVelocity', V, k, u0, f);
41
     b1 = assemble (L1, noslip);
42
     utmp = A1 \setminus b1;
43
     u1 = Function ('u1', V, utmp);
44
45
     # Pressure correction
46
     L2 = LinearForm ('PressureUpdate', Q, u1, k);
47
     [A2, b2] = assemble_system (a2, L2, inflow, outflow);
48
     ptmp = A2 \setminus b2;
49
     p1 = Function ('p1', Q, ptmp);
     # Velocity correction
53
     L3 = LinearForm ('VelocityUpdate', V, k, u1, p1);
     b3 = assemble (L3, noslip);
     ut = A3 \setminus b3;
     u1 = Function ('u0', V, ut);
56
     # Save to file
5.8
     save (p1, sprintf ("p_%3.3d", ++i));
5.9
     save (u1, sprintf ("u_%3.3d", i));
60
     # Move to next time step
62
     u0 = u1;
63
     t += dt
64
   end
```

```
name = [tmpnam ".geo"];

fid = fopen (name, "w");
```

```
fputs (fid, "Point (1) = \{0, 0, 0, 0.1\}; \n"\};
   fputs (fid, "Point (2) = {1, 0, 0, 0.1}; \n");
   fputs (fid, "Point (3) = \{1, 0.5, 0, 0.1\}; \n"\};
fputs (fid, "Point (4) = {0.5, 0.5, 0, 0.1};\n");
   fputs (fid, "Point (5) = \{0.5, 1, 0, 0.1\}; \n"\};
   fputs (fid, "Point (6) = {0, 1, 0,0.1};\n");
   fputs (fid, "Line (1) = \{5, 6\}; \n");
   fputs (fid, "Line (2) = \{2, 3\}; \n");
   fputs (fid, "Line(3) = \{6,1,2\}; n");
13
   fputs (fid, "Line(4) = \{5,4,3\}; \n");
14
   fputs (fid, "Line Loop(7) = \{3,2,-4,1\}; \n"\};
   fputs (fid, "Plane Surface(8) = {7}; \n");
   fclose (fid);
   msho = msh2m_gmsh (canonicalize_file_name (name)(1:end-4),...
                      "scale", 1,"clscale", .2);
   unlink (canonicalize_file_name (name));
```

4.2 A penalization method to take into account obstacles in incompressible viscous flows

# Appendix A

# API reference

## A.1 Import problem defined with ufl

#### import ufl BilinearForm

Function File: = import ufl BilinearForm (myproblem)

Import a BilinearForm from a ufl file.

myproblem is the name of the ufl file where the Bilinear Form is defined.

This function creates in the pwd a file called myproblem\_BilinearForm.oct.

**See also:** import\_ufl\_Problem, FunctionSpace, BilinearForm, LinearForm, Functional.

#### import ufl LinearForm

Function File: = import ufl LinearForm (myproblem)

Import a LinearForm from a ufl file.

myproblem is the name of the ufl file where the Linear Form is defined. This function creates in the pwd a file called myproblem LinearForm.oct.

See also: import\_ufl\_Problem, FunctionSpace, BilinearForm, LinearForm, Functional.

#### import ufl Functional

Function File:  $= import\_ufl\_Functional\ (myproblem)$ 

Import a Functional from a ufl file.

myproblem is the name of the ufl file where the Functional is defined. This function creates in the pwd a file called myproblem Functional.oct.

**See also:** import\_ufl\_Problem, FunctionSpace, BilinearForm, LinearForm, Functional.

#### import ufl FunctionSpace

Function File:  $= import \ ufl \ FunctionSpace \ (myproblem)$ 

Import a FunctionSpace from a ufl file.

myproblem is the name of the ufl file where the FunctionSpace is defined. This function creates in the pwd a file called myproblem FunctionSpace.oct.

See also:  $import\_ufl\_Problem$ , FunctionSpace, BilinearForm, LinearForm, Functional.

#### import ufl Problem

Function File:  $= import\_ufl\_Problem (myproblem)$ 

Import a Variational Problem from a ufl file.

myproblem is the name of the ufl file where the Bilinear Form, the Linear Form and the FunctionSpace are defined.

**See also:** import\_ufl\_BilinearForm, FunctionSpace, BilinearForm, LinearForm, Functional.

## A.2 Problem geometry and FE space

#### Mesh

Function File:  $[mesh \ out] = Mesh \ (mesh \ in)$ 

Construct a mesh from file or from (p, e, t) format. The  $mesh\_in$  should be either

- a string containing the name of the file where the mesh is stored in .xml file If the file is not a .xml file you can try to use the commanddolfin-convert directly from the terminal.
- a a PDE-tool like structure with matrix fields (p,e,t)

The output  $mesh\_out$  is a representation of the  $mesh\_in$  which is compatible with fem-fenics. The easiest way for dealing with meshes is using the msh pkg.

See also: FunctionSpace.

#### **FunctionSpace**

Function File: V = FunctionSpace (myproblem, mesh)

Generate a FunctionSpace on a specific mesh.

This function takes as input the name myproblem of the ufl file where the FunctionSpace is defined and the mesh where it has to be created.

See also: FunctionSpace, SubSpace, import ufl FunctionSpace.

#### SubSpace

Function File: [V1] = SubSpace (V, index)

Extract a SubSpace from an object of type FunctionSpace. The input arguments are

- $\bullet$  V which is a FunctionalSpace
- *index* is a positive integer number which represents the Sub-Space which has to be extracted.

The output V1 is the SubSpace needed.

See also: FunctionSpace.

#### A.3 Problem variables

#### Constant

Function File: [c] = Constant (name, value)

Creates a constatnt object over all the mesh elements with the value specified.

This function take as input the *name* of the Constant that has to be created and its *value*, which can be either a scalar or a vector.

See also: Expression, Function.

#### Expression

Function File:  $[f] = Expression (name, Function\_handle)$ 

Creates an object with the value specified as a function handle. The input parameters are

- name is the name of the coefficient as it is declared in the uff file
- Function\_handle is a function handle which specify the expression to apply for our coefficient

The output f is an object which contains a representation of the function

See also: Constant, Function.

#### **Function**

Function File: [func] = Function (name, FunctionSpace (or Function), Vector (or index))

Initialize an object with the values specified in a vector or extracting a component from a vectorial field. This function can be used in two different ways

- To create a function from a vector. In this case, the arguments are:
  - name is a string representing the name of the function
  - FunctionSpace is the fem-fenics function space where the vector is defined
  - *Vector* specifies the values of the coefficients for each basisc function of the FunctioSpace
- To extract a scalar field from a vectorial one
  - name is a string representing the name of the function
  - Function is the vector valued Function
  - Index contains the index of the scalar field to extract. Index starts from 1.

The output *func* is an object which contains a representation of the function *Vector* which can be plotted or saved or passed as argument for a variational problem.

See also: Constant, Expression, plot, save.

#### DirichletBC

Function File: [bc] = Dirichlet BC (Function Space, Boundary\_Label, Function handle)

Specify essential boundary condition on a specific side. The input parameters are

- $\bullet$  FunctionSpace is the fem-fenics space where we want to apply the BC
- Function\_handle is a function handle which contains the expression that we want to apply as a BC. If we have a Vector field, we can just use a vector of function handles: Function handle = [@(x, y) f1, @(x, y) f2, ...]
- Boundary\_Label is an Array which contains the label(s) of the side(s) where the BC has to be applied.

The output bc is an object which contains the boundary conditions **See also:** Mesh, FunctionSpace.

## A.4 Definition of the abstract Variational problem

#### BilinearForm

Function File:  $[a] = BilinearForm \ (my\_problem, U, V, coefficient\_1, coefficient 2,...)$ 

Construct a BilinearForm previously imported from ufl. The compulsory arguments are:

- my problem the name of the problem to solve.
- the FunctionSpace U and V where the problem is defined.

The optional arguments are the *coefficient\_1*, *coefficient\_2* which specify the parameters for the BilinearForm as stated in the ufl file. They can be either a Constant, a Function or an Expression.

**See also:** import\_ufl\_BilinearForm, import\_ufl\_Problem, FunctionSpace, LinearForm, ResidualForm.

#### LinearForm

Function File:  $[L] = LinearForm (my\_problem, U, coefficient\_1, coefficient\_2,...)$ 

Construct a Functional previously imported from a ufl file. The compulsory arguments are:

- $\bullet \ my\_problem$  the name of the problem to solve.
- $\bullet$  the FunctionSpace U where the problem is defined.

The optional arguments are the *coefficient\_1*, *coefficient\_2* which specify the parameters for the LinearForm with the same name which was used in the uff file. They can be either a Constant, a Function or an Expression.

**See also:** import\_ufl\_LinearForm, import\_ufl\_Problem, BilinearForm, ResidualForm, BilinearForm.

#### ResidualForm

Function File:  $[L] = LinearForm \ (my\_problem, \ U, \ coefficient\_1, \ coefficient\_2,...)$ 

Construct a Residual Form previously imported from a ufl file with the function import ufl Linear Form.

The compulsory arguments are:

- my problem the name of the problem to solve.
- $\bullet$  the FunctionSpace U where the problem is defined.

The optional arguments are the *coefficient\_1*, *coefficient\_2* which specify the parameters for the ResidualForm with the same name which was used in the uff file. They can be either a Constant, a Function or an Expression.

**See also:** import\_ufl\_LinearForm, import\_ufl\_Problem, BilinearForm, ResidualForm, BilinearForm.

#### JacobianForm

Function File:  $[J] = Functional \ (my\_problem, \ U, \ V, \ coefficient\_1, \ coefficient \ 2,...)$ 

Construct a JacobianForm previously imported from a ufl file with the function import ufl BilinearForm.

The compulsory arguments are:

- my problem the name of the problem to solve.
- the FunctionSpace U and V where the problem is defined.

The optional arguments are the *coefficient\_1*, *coefficient\_2* which specify the parameters for the JacobianForm with the same name which was used in the uff file. They can be either a Constant, a Function or an Expression.

 $\bf See\ also:\ import\_ufl\_BilinearForm,\ LinearForm,\ ResidualForm,\ BilinearForm.$ 

#### **Functional**

Function File: [L] = Functional (my problem, U, coefficient 1, coefficient 2,...)

Construct a Functional previously imported from a ufl file. The compulsory arguments are:

- my problem the name of the problem to solve.
- $\bullet$  the FunctionSpace U where the problem is defined.

The optional arguments are the *coefficient\_1*, *coefficient\_2* which specify the parameters for the Functional with the same name which was used in the ufl file. They can be either a Constant, a Function or an Expression.

See also:  $import\_ufl\_Functional$ , LinearForm, ResidualForm, BilinearForm.

## A.5 Creation of the discretized problem

#### assemble

Function File: [A],  $[x(Optional)] = assemble (form_a, Dirichlet BC)$ 

Construct the discretization of a Form and apply essential BC. The input arguments are

- form\_a which is the form to assemble. It can be a form of rank 2 (BilinearForm or JacobianForm), a form of rank 1 (LinearForm or ResidualForm) or a form of rank 0 (Functional).
- Dirichlet BC represents the optional BC applied to the system.

The output A is a discretized representation of the form a:

- A is a sparse Matrix if form a is a bilinear form
- A is a Vector if form a is a linear form
- A is a Double if form a is a functional

If boundary condition has to be applied to a vector for a nonlinear problem then it should be provide as 2nd argument and it will be given back as the second output argument. For an example of this situation, please refer to the HyperElasticity example.

**See also:** BilinearForm, LinearForm, ResidualForm, JacobianForm, Functional.

#### assemble system

Function File: [A], [b],  $[x(Optional)] = assemble\_system (form\_a, form\_L, DirichletBC)$ 

Construct the discretization of a system and apply essential BC. The input arguments are

- form a which is the BilinearForm to assemble.
- form L which is the LinearForm to assemble.
- DirichletBC represents the optional BC applied to the system.

The output A is a matrix representing the  $form\_a$  while b represents  $form\_L$ . If boundary conditions have to be applied to a vector for a nonlinear problem then it should be provide as 3rd argument and it will be given back as the 3rd output argument. For an example of this situation, please refer to the HyperElasticity example.

**See also:** BilinearForm, LinearForm, ResidualForm, JacobianForm, Functional.

## A.6 Post processing

#### @function/save

Function File: fem\_save (Function, Name)

Save a function in vtu format. The input parameters are

- Function is the function that you want to save
- Name is a string for the output name

The output is a file in format .vtu See also: plot, Function.

#### @function/plot

Function File: plot (Function)

Plot a Function.

See also: Function, Save.

#### @mesh/plot

Function File: plot (Mesh, Nodal Values(OPTIONAL))

Plot a Mesh. The input parameter is the Mesh and optionally also a vector representing the values of a function at each node. **See also:** Mesh, save.

#### @function/feval

Function File: [value] = feval (function name, Coordinate)

Evaluate a function at a specific point of the domain and return the value. The input parameters are the function and the point where it has tobe evaluated.

See also: Function.

## Appendix B

# Autoconf and Automake

In this section we want to discuss how we can write a config.ac and a Makefile.in files which:

- check if a program is available and stop if it is not
- check if a header file is available and issue a warning if not, but go ahead with the compilation

To reach this goal, we need two components:

**configure.ac** Is a file which checks whether the program/header is available or not and sets consequently the values of some variables.

```
# Checks if the program mkoctfile is available and sets the variable
    HAVE_MKOCTFILE consequently
AC_CHECK_PROG([HAVE_MKOCTFILE], [mkoctfile], [yes], [no])
# if mkoctfile is not available, it issues an error and stops the
    compilation
if [test $HAVE_MKOCTFILE = "no"]; then
 AC_MSG_ERROR([mkoctfile required to install $PACKAGE_NAME])
#Checks if the header dolfin.h is available; if it is available, the
    value of the ac_dolfin_cpp_flags is substituted with
    -DHAVE_DOLFIN_H, otherwise it is left empty and a warning
    message is printed
AC_CHECK_HEADER([dolfin.h],
  [AC_SUBST(ac_dolfin_cpp_flags,-DHAVE_DOLFIN_H)
      AC_SUBST(ac_dolfin_ld_flags,-ldolfin)],
  [AC_MSG_WARN([dolfin headers could not be found, some
      functionalities will be disabled, don't worry your package
      will still be working, though.])] ).
# It generates the Makefile, using the template described below
AC_CONFIG_FILES([Makefile])
```

Makefile.ac This file is a template for the Makefile, which will be automatically generated when the configure.ac file is executed. The values of the variable ac\_dolfin\_cpp\_flags and ac\_dolfin\_ld\_flags are substituted with the results obtained above:

```
CPPFLAGS += @ac_dolfin_cpp_flags@
LDFLAGS += @ac_dolfin_ld_flags@
```

In this way, if dolfin.h is available, CPPFLAGS contains also the flag -  $\operatorname{DHAVE}\ \operatorname{DOLFIN}\ \operatorname{H}.$ 

program.cc Our .cc program, should thus include the header dolfin.h only if -DHAVE\_DOLFIN\_H is defined at compilation time. For example

```
#ifdef HAVE_DOLFIN_H
#include <dolfin.h>
#endif
int main ()
{

#ifndef HAVE_DOLFIN_H
error("program: the program was built without support for dolfin");

#else
/* Body of your function */
#endif
return 0;
}
```

Warning If in the Makefile.in you write something like

```
HAVE_DOLFIN_H = @HAVE_DOLFIN_H@

ifdef HAVE_DOLFIN_H

CPPFLAGS += -DHAVE_DOLFIN_H

LIBS += -ldolfin

endif
```

it doesn't work because the variable HAVE\_DOLFIN\_H seems to be always defined, even if the header is not available.

# Bibliography

- [1] http://www.gnu.org/software/octave/download.html.
- [2] http://fenicsproject.org/download/.
- [3] http://fenicsproject.org/documentation/ufl/1.2.0/user/user\_manual.html.
- [4] http://octave.sourceforge.net/msh/index.html.
- [5] http://octave.sourceforge.net/fpl/index.html.
- [6] http://octave.sourceforge.net/secs1d/index.html.
- [7] http://octave.sourceforge.net/bim/index.html.
- [8] http://jordi.platinum.linux.pl/octave/what-is-octave.pdf.
- [9] Luca Formaggia. Advanced programming for scientific computing. lecture title: Smart pointers. 2012.
- [10] http://fenicsproject.org/documentation/dolfin/1.2.0/cpp/programmers-reference/mesh/MeshFunction.html.
- [11] http://fenicsproject.org/documentation/dolfin/1.2.0/cpp/programmers-reference/mesh/MeshValueCollection.html.
- [12] http://fenicsproject.org/documentation/dolfin/1.2.0/cpp/programmers-reference/mesh/MeshDomain.html.
- [13] http://fenicsproject.org/documentation/dolfin/1.2.0/cpp/programmers-reference/mesh/MeshData.html.
- [14] http://gedeone-gsoc.blogspot.co.uk/2013/06/update-4.html.
- [15] Anders Logg, Kent-Andre Mardal, and Garth Wells. Automated solution of differential equations by the finite element method: The fenics book, volume 84. Springer, 2012.